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Study and Development Of Some Applications Of Toeplitz and Hankel Matrices

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Abstract

In recent years, there has been surge of applications of Toeplitz and Hankel matrices in various fields of Science and Engineering. With increased computing power available over last 2 decades, research on numerical methods in this regards has also advanced. The aim of this research is to analyze and extend further mathematical and computational methods (to obtain inverses, eigen values, solving system of equations) as also to develop new and more effective techniques to further applications of the said matrices to illuminate information theory and coding, and security.

Key words

Toeplitz matrices, Hankel matrices, information theory, security

1) Introduction

a. What are Toeplitz, Hankel and Circulant Matrices?

i) **Toeplitz matrix**:[1,2] In linear algebra, a **Toeplitz matrix** or **diagonal-constant matrix**, named after Otto Toeplitz, is a matrix in which each descending diagonal from left to right is constant.

For instance, the following matrix is a Toeplitz matrix:

$$\begin{bmatrix} a & b & c & d & e \\ f & a & b & c & d \\ g & f & a & b & c \\ h & g & f & a & b \\ i & h & g & f & a \end{bmatrix}.$$

Any $n \times n$ matrix A of the form

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \dots & \dots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \dots & \dots & a_2 & a_1 & a_0 \end{bmatrix}$$

is a **Toeplitz matrix**. If the i,j^{th} element of A is denoted $A_{i,j}$, then we have $A_{i,j} = A_{i+1,j+1} = a_{i-j}$

ii) **Hankel matrix [3]**: In linear algebra, a **Hankel matrix** (or **catalecticant matrix**), named after Hermann Hankel, is a square matrix in which each ascending skew-diagonal from left to right is constant.
For instance, the following matrix is a Hankel matrix:

$$\begin{bmatrix} a & b & c & d & e \\ b & c & d & e & f \\ c & d & e & f & g \\ d & e & f & g & h \\ e & f & g & h & i \end{bmatrix}.$$

Any $n \times n$ matrix A of the form

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ a_1 & a_2 & & & & \vdots \\ a_2 & & & & & \vdots \\ \vdots & & & & & a_{2n-4} \\ \vdots & & & & a_{2n-4} & a_{2n-3} \\ a_{n-1} & \cdots & \cdots & a_{2n-4} & a_{2n-3} & a_{2n-2} \end{bmatrix}$$

is a **Hankel matrix**. If the i,j^{th} element of A is denoted $A_{i,j}$, then we have $A_{i,j} = A_{i+1,j-1} = a_{i+j-2}$

The Hankel matrix is closely related to the Toeplitz matrix (a Hankel matrix is an upside-down Toeplitz matrix).

iii) **Circulant matrix[4]:** A common special case of Toeplitz matrices — which will result in significant simplification and play a fundamental role in developing more general results — results when every row of the matrix is a right cyclic shift of the row above it so that $c_k = c_{-(n-k)} = c_{k-n}$ for k = 1, 2,... , n − 1.

An n x n circulant matrix C takes the form

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}.$$

A circulant matrix is fully specified by one vector, c , which appears as the first column of C. The remaining columns of C are each cyclic permutations of the vector C with offset equal to the column index. The last row of C is the vector c in reverse order, and the remaining rows are each cyclic permutations of the last row.

b. Immediate role in Linear Algebra and Information Theory

a) A matrix equation of the form $\boldsymbol{Ax=b}$ is called a **Toeplitz system** if A is a Toeplitz matrix. If A is an **n x n** Toeplitz matrix, then the system has only $2n-1$ degrees of freedom, rather than n^2. We might therefore expect that the solution of a Toeplitz system would be easier, and indeed that is the case.

b) Circulant matrices arise, for example, in applications involving the discrete Fourier transform (DFT) and the study of **cyclic codes for error correction**.

2) Review of Literature and Development in the subject :

Toeplitz matrices are ubiquitous and one of the most well-studied and understood classes of structured matrices.

They arise in pure mathematics: algebra, algebraic geometry, analysis, combinatorics, differential geometry, graph theory, integral equations, operator algebra, partial differential equations, probability, representation theory, topology; as well as in applied mathematics: approximation theory, compressive sensing numerical integral equations, numerical integration, numerical partial differential equations, image processing, optimal control quantum mechanics, queuing networks, signal processing, statistics, time series analysis, among other areas.

Secondly, Toeplitz matrices have some of the most attractive computational properties and are amenable to a wide range of disparate algorithms.

Few of the most important results concerning these matrices are enlisted below:

1) **a) Every n×n matrix is generically a product of ⌊n/2⌋+1 Toeplitz matrices and always a product of at most 2n + 5 Toeplitz matrices. [6]**

   b) The same result holds true if the word 'Toeplitz' is replaced by 'Hankel', and the generic bound ⌊n/2⌋ +1 is sharp. [6]

2) **Given a sequence of Toeplitz matrices, there exists asymptotically equivalent sequences of circulant matrices [7]**

3) **Assume R is a nonsingular n x n matrix whose diagonal elements R_{22} , ..., $R_{n-1\ n-1}$ are nonzero. The matrix R has the triangle property if and only if its inverse A is tridiagonal. [8]**

   Suppose R is an n x n matrix whose diagonal elements R_{22}, R_{33}, ... , $R_{n-1\ n-1}$ are nonzero. We say that R has the triangle property if $R_{ij} = \frac{R_{ik} R_{kj}}{R_{kk}}$ for all $i < k < j$ and for all $i > k > j$

4) **The Vandermonde decomposition is a classical result by Caratheodory and Fej´er dating back to 1911 . If T is an n × n positive semidefinite (PSD) Toeplitz matrix of rank r < n, then the result states that T can be uniquely decomposed as $T = APA^H$ where P is an r × r positive definite diagonal matrix and A is an n × r Vandermonde matrix whose columns correspond to uniformly sampled complex sinusoids with different frequencies, A^H denote the Hermitian transpose of the matrix A. The result has become important in the area of data analysis and signal processing. [9]**

3) **Objectives of Research/ Proposed Hypothesis:**

To explore/develop various mathematical results /computational techniques concerning the field of 'Toeplitz and Hankel matrices applications' in following areas (but not limited to)

i) **Information theory and coding**
ii) **Security in cloud computing**
iii) **Biometric security**
iv) **Numerical methods for inverses, Eigen values of Toeplitz and Hankel matrices.**

4) **Methodology to be adopted:**

It is proposed to use a mixed method research approach which includes qualitative and quantitative strategies that involve compiling, studying and analyzing techniques currently in use and developing/ suggesting further techniques to use for potential situations arising in future. Action research attempts to create new knowledge from the specific contexts through investigation along the progress [10]. It starts with deliberating about current circumstances then leads to improved processes and outcomes [11]. The research will include evaluation of methods-cum-techniques currently in use and development of the framework for possible future scenario. The work may also entail developing new mathematical software in this context.

5) Importance of study/ Society application:

a. International / National status

Almost all over the world the mathematical filed of matrices, in general, is being used to study each and every branch of Science, Technology and Finance. Toplitz and Hankel matrices, in particular, have been excessively used in the past few decades in many of the sub-fields of applied mathematics, engineering, theory of finance, etc.

b. Significance of the study

Apparently the subject matter seems to be a narrow field, however, the fast pace with which various results, techniques and ideas that have developed in the recent times open up the rich treasures of scientific and mathematical knowledge where the proposed research may bring about still new insights and development of new/ more efficient algorithms.

6) Proposed work Plan/ Formulation and Structure of Study:

a. Year-wise Plan of work and targets to be achieved

Year	**:**	**Proposed work plan**
First	:	Exhaustive literature review and critiquing of present status of results and techniques in context of the proposed research topic.
Second	:	Developing insight into possible applications of the said type of matrices in various aspects of "Information Theory and Coding"
Third	:	Developing comparatively more efficient algorithms and computational techniques then those currently in vogue for obtaining inverses, eigen values of the aforementioned matrices.

The aforementioned work in its entirety is expected to lead to simplification in the ways in which these matrices are found useful in ever increasing number of areas.

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